Theoretical Computer Science 127 (1994) 387–393
Elsevier

Note

Hyperedge channels are abelian

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Communicated by A. Salomaa
Received June 1993

Abstract

Deutz, A.H., A. Ehrenfeucht and G. Rozenberg, Hyperedge channels are abelian, Theoretical Computer Science 127 (1994) 387–393.

In this note we generalize the notion of a *bidirectional* communication channel as used in dynamic labeled 2-structures to a hyperedge channel which involves *many* participants. We prove that the communication in such a channel is governed by an abelian group.

0. Preliminaries

In the sequel we use standard notation. In particular, for a set X we use X^X to denote the set of all functions from X into X; $|X|$ denotes the cardinality of X, and id_X denotes the identity function on X.

We assume the reader to be familiar with the basic notions from group theory and graph theory; see, e.g., [3, 1].

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SSDI 0304-3975(93)E0157-Y

1. Mutating schemes

In [2] mutating schemes have been studied. They serve, e.g., as a model for a network of processors. In this model each node x is a processor of the network and the actions of such a processor consist of output actions O_x and input actions I_x. In a given global state of a network all kinds of relationships hold between nodes of the network. Hence a global state is represented by a directed edge-labeled graph g and the set of processors is represented by the set of nodes of g. For each edge (x, y), its label, $lab_g(x, y) = b$, says that b is the relationship between x and y in this global state g. Then, when an output action $\psi \in O_x$ takes place in x it will affect the relationships between x and the other nodes by changing the label $lab_g(x, y)$ of each outgoing edge (x, y) to $\psi(lab_g(x, y))$. Analogously, an input action $\eta \in I_x$ will change the label $lab_g(y, x)$ of each incoming edge (y, x) to $\eta(lab_g(y, x))$.

Formally a mutating scheme is defined as follows.

Definition 1.1. A *mutating scheme* is a 4-tuple $K = (D, \Delta, O, I)$, where D and Δ are nonempty sets, $O = \{O_x \mid x \in D\}$, $I = \{I_x \mid x \in D\}$, where for each $x \in D$, $O_x \subseteq \Delta^\Delta$ and $I_x \subseteq \Delta^\Delta$.

Simply transitive mutating schemes are mutating schemes satisfying the following four axioms.

(A1) For each $x \in D$, O_x and I_x are closed under composition.

(A2) For all $a, b \in \Delta$ and each $x \in D$, there exist $\varphi \in O_x$ and $\psi \in I_x$ such that $\varphi(a) = b$ and $\psi(a) = b$.

(A3) For all $x, y \in D$ such that $x \neq y$, $\varphi \in O_x$, $\gamma \in I_y$, and $a \in \Delta$, $\varphi(\gamma(a)) = \gamma(\varphi(a))$.

(A4) $|D| \geqslant 3$.

The axioms (A1)–(A3) have a rather clear intuition. Axiom (A1) says that each change of labels outgoing from x (incoming to x) that can be achieved by applying a composition of functions from O_x (I_x) can be achieved in "one stroke" by using one output (input) function in x. Axiom (A2) says that if one wishes to change a specific label a outgoing from x (incoming to x) into a specific label b outgoing from x (incoming to x) then it is possible to do it by appropriately choosing an output (input) function in x. Axiom (A3) allows applying output and input functions concurrently throughout any graph spanned on D. Axiom (A4) is rather technical – its usefulness is discussed in [2].

We will use the following facts from [2] about simply transitive mutating schemes.

Proposition 1.2. *Let $K = (D, \Delta, O, I)$ be a simply transitive mutating scheme. Let $x \in D$ and let $\Psi \in \{O_x, I_x\}$.*

(1) *For all $\varphi \in \Psi$, if $\varphi(a) = a$ for some $a \in \Delta$, then $\varphi = id_\Delta$.*

(2) *For all $\psi_1, \psi_2 \in \Psi$, if $\psi_1(a) = \psi_2(a)$ for some $a \in \Delta$, then $\psi_1 = \psi_2$.*

Proposition 1.3. *Let $K=(D,\Delta,O,I)$ be a simply transitive mutating scheme. For all $x,y\in D$, $O_x=O_y$ and $I_x=I_y$.*

In view of Proposition 1.3 we denote the common sets O_x (I_x) of a simply transitive mutating scheme $K=(D,\Delta,O,I)$ by O_k (I_k), respectively.

If we require that $|D|=2$ (hence, we abolish axiom (A4)) then the above proposition does not hold. It is shown in [2] that in this case the situation is rather arbitrary in the sense that the work of such a bidirectional communication channel can be described using two groups, one "governing" the edge (x,y) and the other "governing" the edge (y,x), where $D=\{x,y\}$.

2. Hyperedge channels

In this note we will consider communication channels which consist of more than two participants. It turns out that the situation for such channels is not arbitrary anymore. The work of the whole channel is "governed" by an abelian group.

Definition 2.1. A *hyperedge channel* is a triple $H=(D,\Delta,F)$, where D is a finite nonempty set, Δ is a nonempty set, $F=\{F_x\,|\,x\in D\}$, where for each $x\in D$, $F_x\subseteq\Delta^\Delta$.

The set D is called the *hyperedge* of H or the *domain of* H, denoted by $dom(H)$, and Δ is called the *alphabet of* H, denoted by $alph(H)$. For each node x of the hyperedge D, F_x is the *set of (input–output) functions in* x.

Analogous to simply transitive mutating schemes, we define *simply transitive hyperedge channels* to be channels satisfying the following four axioms. (Below, the elements of D are denoted by $x_1,x_2,\ldots,x_n$, where n is the cardinality of D.)

(H1) For each i such that $1\leqslant i\leqslant n$, F_{x_i} is closed under composition.

(H2) For all $a,b\in\Delta$, and each i such that $1\leqslant i\leqslant n$, there exists a $\varphi\in F_{x_i}$ such that $\varphi(a)=b$.

(H3) For all $\varphi_1\in F_{x_1}$, $\varphi_2\in F_{x_2},\ldots,\varphi_n\in F_{x_n}$, for all $a\in\Delta$, and for all permutations π on $\{1,2,\ldots,n\}$, $\varphi_1\varphi_2\ldots\varphi_n(a)=\varphi_{\pi(1)}\varphi_{\pi(2)}\cdots\varphi_{\pi(n)}(a)$.

(H4) $|D|\geqslant 3$.

Note that for $n=2$ we have a communication channel with 2 participants as in the mutating scheme model, except that $O_x=I_x=F_x$, for both nodes x of the channel.

The following lemma gives the basic technical property of simply transitive hyperedge channels.

Lemma 2.2. *Let $H=(D,\Delta,F)$ be a simply transitive hyperedge channel and let $n=|D|$. For each i such that $1\leqslant i\leqslant n$, $id_\Delta\in F_{x_i}$.*

Proof. Choose an arbitrary $i_1\in\{1,2,\ldots,n\}$ and an arbitrary $a\in\Delta$. By axiom (H2) we can choose a transformation $\varphi_{i_1}\in F_{x_{i_1}}$ such that $\varphi_{i_1}(a)=a$. We will show that for each

$b\in\Delta$, $\varphi_{i_1}(b)=b$. Assume $b\in\Delta$. Choose an $i_2\in\{1,2,\dots,n\}$ such that $i_1\neq i_2$. By axiom (H2) there exists a $\varphi_{i_2}\in F_{x_2}$ such that $\varphi_{i_2}(a)=b$. Furthermore, by axiom (H2), for each $j\in\{1,2,\dots,n\}\setminus\{i_1,i_2\}$, there exists a transformation $\varphi_j\in F_{x_j}$ such that $\varphi_j(a)=a$. Let $j_1,\dots,j_{n-2}$ be such that $\{j_1,j_2,\dots,j_{n-2}\}=\{1,2,\dots,n\}\setminus\{i_1,i_2\}$. Then by axiom (H3) $\varphi_{i_1}\varphi_{i_2}\varphi_{j_1}\cdots\varphi_{j_{n-2}}(a)=\varphi_{i_2}\varphi_{i_1}\varphi_{j_1}\cdots\varphi_{j_{n-2}}(a)$. From this we get by our choice of functions, $\varphi_{i_1}\varphi_{i_2}(a)=\varphi_{i_2}\varphi_{i_1}(a)$. Hence, $\varphi_{i_1}(b)=b$. Since b was an arbitrary choice, the assumption $\varphi_{i_1}(a)=a$ implies that, for each $b\in\Delta$, $\varphi_{i_1}(b)=b$ and so $\varphi_{i_1}=id_\Delta$. Hence, $id_\Delta\in F_{x_{i_1}}$ and therefore, since i_1 was an arbitrary choice, $id_\Delta\in F_{x_i}$, for each $i\in\{1,2,\dots,n\}$. □

We establish now the relationship between simply transitive hyperedge channels and simply transitive mutating schemes. Simply transitive hyperedge channels correspond to simply transitive mutating schemes for which at each node the set of input functions is equal to the set of output functions. The crucial property to establish this is the fact that for any subset of transformations $S=\{\eta_1,\eta_2,\dots,\eta_s\}$ of Δ^Δ with $s\leqslant n$ such that for each $i\in\{1,2,\dots,n\}$, $S\cap F_{x_i}$ contains at most one element, $\eta_1\eta_2\cdots\eta_s=\eta_{\pi(1)}\eta_{\pi(2)}\cdots\eta_{\pi(s)}$, where π is any permutation of $\{1,2,\dots,s\}$. (This property follows easily from Lemma 2.2.)

Lemma 2.3. *A triple $H=(D,\Delta,F)$ is a simply transitive hyperedge channel iff the 4-tuple $K=(D,\Delta,F,F)$ is a simply transitive mutating scheme.*

Proof. For the only-if-part of the statement of the lemma we proceed as follows. Let $H=(D,\Delta,F)$ be a simply transitive hyperedge channel. Clearly, $K=(D,\Delta,F,F)$ satisfies axioms (A1), (A2), and (A4). In order to show that K satisfies (A3) we proceed as follows. Let $n=|D|$. Assume $\varphi\in F_{x_{i_2}}$ and $\psi\in F_{x_{i_2}}$, for some $i_1,i_2\in\{1,2,\dots,n\}$ such that $i_1\neq i_2$. We wish to show that $\varphi\psi=\psi\varphi$. By Lemma 2.2 and axiom (H3) $\varphi\,\psi\,id_\Delta\dots id_\Delta=\psi\varphi\,id_\Delta\dots id_\Delta$ with $n-2$ id_Δ's and so $\varphi\psi=\psi\varphi$.

For the if-part of the statement of the lemma we assume that $K=(D,\Delta,F,F)$ satisfies axioms (A1)–(A4). Clearly, axioms (H1), (H2), and (H4) are satisfied by $H=(D,\Delta,F)$. In order to show that (H3) is satisfied we assume that $\varphi_i\in F_{x_i}$, $i=1,\dots,n$. Since by axiom (A3), for any pair φ_i,φ_j such that $i\neq j$, $\varphi_i\varphi_j=\varphi_j\varphi_i$, we get that $\varphi_1\varphi_2\cdots\varphi_n=\varphi_{\pi(1)}\varphi_{\pi(2)}\cdots\varphi_{\pi(n)}$, for any permutation π of $\{1,2,\dots,n\}$. Hence, (H3) is satisfied. □

Following [2] we introduce now a group structure on the alphabet Δ. Let a_0 be an arbitrary element of Δ which we will fix for the sequel. (In general different elements of Δ which are fixed will give rise to different, but isomorphic group structures on Δ; see Remark 2.7.) Let for each $a\in\Delta$ and each $i\in\{1,2,\dots,n\}$ (where n is the cardinality of the hyperedge), φ_{a,x_i} be a function $\varphi\in F_{x_i}$ such that $\varphi(a_0)=a$. Now let us choose an $x_0\in D$ and let $\circ$ be the operation defined by

$$a\circ b=\varphi_{a,x_0}\varphi_{b,x_0}(a_0)=\varphi_{a,x_0}(b),\quad \text{where } a,b\in\Delta.$$

In the following lemma we will show that the functions φ_{a,x_i} defined above are unique and that the above definition of $\circ$ does not depend on the choice of x_0.

Lemma 2.4. *Let $H=(D,\Delta,F)$ be a simply transitive hyperedge channel. Let $\varphi\in F_{x_i}$ and $\psi\in F_{x_j}$, for some $i,j\in\{1,2,\dots,n\}$ (including the case $i=j$). If, for some $a\in\Delta$, $\varphi(a)=\psi(a)$, then $\varphi=\psi$.*

Proof. Let $H=(D,\Delta,F)$ be a simply transitive hyperedge channel. Let $\varphi\in F_{x_i}$ for some $i\in\{1,2,\dots,n\}$ and let $\psi\in F_{x_j}$ for some $j\in\{1,2,\dots,n\}$. Suppose there exists an element a in Δ such that $\varphi(a)=\psi(a)$. By Lemma 2.3 $K=(D,\Delta,F,F)$ is a simply transitive mutating scheme. Hence, by Proposition 1.3 $F_{x_i}=F_{x_j}$. Now we can apply Proposition 1.2 to get $\varphi=\psi$. □

The following lemma gives a condition on a simply transitive mutating scheme under which the induced group $(\Delta,\circ)$ is abelian.

Lemma 2.5. *Let $K=(D,\Delta,O,I)$ be a simply transitive mutating scheme such that $O_K=I_K$. Let a_0 be a fixed element in Δ. Define, for all $a,b\in\Delta$, $a\circ b=\varphi_a(b)$, where φ_a is the unique element in O_K such that $\varphi_a(a_0)=a$. Then $(\Delta,\circ)$ is an abelian group.*

Proof. Let $K=(D,\Delta,O,I)$ be a simply transitive scheme such that $O_x=I_x$, for all $x\in D$ (or equivalently, $O_K=I_K$). First of all note that by Proposition 1.2(2) the functions φ_a defined in the lemma are indeed unique. Secondly, as stated in [2], let us notice that for each $d\in\Delta$ and for each $\varphi\in O_K$, $\varphi(d)=\varphi(a_0)\circ d$. This is seen as follows. Let $\varphi_{\varphi(a_0)}$ be the function $\kappa\in O_K$ such that $\kappa(a_0)=\varphi(a_0)$. By Proposition 1.2(2), $\varphi=\varphi_{\varphi(a_0)}$. By the definition of $\circ$ and the fact $\varphi=\varphi_{\varphi(a_0)}$, $\varphi(a_0)\circ d=\varphi_{\varphi(a_0)}\varphi_d(a_0)=\varphi_{\varphi(a_0)}(d)=\varphi(d)$. Hence, $\varphi(d)=\varphi(a_0)\circ d$.

Let a_0 be fixed element in Δ and let $b,c\in\Delta$. Choose $\varphi_1,\varphi_2\in I_K$ such that $\varphi_1(a_0)=b$ and $\varphi_2(a_0)=c$. By axiom (A3), Proposition 1.3, and the fact that $I_K=O_K$, $\varphi_1\varphi_2(a_0)=\varphi_2\varphi_1(a_0)$. So we also have $\varphi_1\varphi_2(a_0)=\varphi_1(a_0)\circ\varphi_2(a_0)=b\circ c$ and $\varphi_2\varphi_1(a_0)=\varphi_2(a_0)\circ\varphi_1(a_0)=c\circ b$. Hence $b\circ c=c\circ b$. Hence, $(\Delta,\circ)$ is an abelian group. □

We now show that simply transitive hyperedge channels are abelian.

Theorem 2.6. *Let $H=(D,\Delta,F)$ be a simply transitive hyperedge channel. Let a_0 be some fixed element of Δ. Choose an element x_0 in D. Define, for all $a,b\in\Delta$, $a\circ b=\varphi_{a,x_0}(b)$, where φ_{a,x_0} is the unique element η in F_{x_0} such that $\eta(a_0)=a$. Then $(\Delta,\circ)$ is an abelian group.*

Proof. Let $H=(D,\Delta,F)$ be a simply transitive hyperedge channel. Note that by Lemma 2.4, the functions φ_{a,x_0} defined in the theorem are indeed unique. Let a_0 be a fixed element in Δ. The definition of $\circ$ does not depend on the choice of x_0 in view of

Lemma 2.4. By Lemma 2.3 we get that $K=(D,\Delta,F,F)$ is a simply transitive mutating scheme. Clearly, we can for the definition of the group structure on Δ replace H by K. By Lemma 2.5 $(\Delta,\circ)$ is an abelian group. $\square$

We conclude this note by the following observations.

Remark 2.7. Let $H=(D,\Delta,F)$ be a simply transitive hyperedge channel. By Lemma 2.3 and Proposition 1.3, for all $x,y\in D$, $F_x=F_y$. Denote this common group by F. For different choices of $a_0\in\Delta$ we get, in general, different, but isomorphic group laws on Δ.

Let a_0 be fixed element in Δ. We can define a group operation $\circ_1$ on Δ as before. Let $a,b\in\Delta$. Then $a\circ_1 b=\varphi_a\varphi_b(a_0)$, where φ_a is the unique element $\psi\in F$ such that $\psi(a_0)=a$ and φ_b is the unique element $\psi'\in F$ such that $\psi'(a_0)=b$.

Let $\bar{a}_0$ be another element in Δ. We define now a second group operation $\circ_2$ on Δ in a similar way: $a\circ_2 b=\bar{\varphi}_a\bar{\varphi}_b(\bar{a}_0)$, where $\bar{\varphi}_a$ is the unique element $\bar{\psi}\in F$ such that $\bar{\psi}(\bar{a}_0)=a$ and $\bar{\varphi}_b$ is the unique element $\bar{\psi}'\in F$ such that $\bar{\psi}'(\bar{a}_0)=b$.

In general the groups $(\Delta,\circ_1)$ and $(\Delta,\circ_2)$ are isomorphic, but not equal. There is a simple relationship between the operations $\circ_1$ and $\circ_2$. Let φ be the unique element in F such that $\varphi(a_0)=\bar{a}_0$. Let c be an arbitrary element Δ, φ_c be the unique element κ in F such that $\kappa(a_0)=c$, and $\bar{\varphi}_c$ the unique element κ' in F such that $\kappa'(a_0)=c$. Since $\bar{\varphi}_c\varphi(a_0)=\varphi_c(a_0)$, we get $\bar{\varphi}_c\varphi=\varphi_c$ and $\bar{\varphi}_c=\varphi_c\varphi^{-1}$. From this we get that $a\circ_1 b=\bar{\varphi}_a\varphi(b)$. We can now contrast this with $a\circ_2 b=\bar{\varphi}_a\bar{\varphi}_b(\bar{a}_0)=\bar{\varphi}_a(b)$ and convince ourselves that in general $\circ_1\neq\circ_2$.

We now proceed to show that the groups $(\Delta,\circ_1)$ and $(\Delta,\circ_2)$ are isomorphic. In fact, φ is an isomorphism of these groups. By the definition of the mappings involved we get that $\varphi\varphi_a(a_0)=\varphi_{\varphi(a)}(a_0)$. Hence, $\varphi\varphi_a=\varphi_{\varphi(a)}$ and so $\varphi\varphi_a(b)=\varphi_{\varphi(a)}(b)$. On the one hand, $\varphi\varphi_a(b)=\varphi(a\circ_1 b)$. On the other hand, $\varphi_{\varphi(a)}(b)=\varphi_{\varphi(a)}\varphi^{-1}\varphi(b)=\bar{\varphi}_{\varphi(a)}(\varphi(b))=\bar{\varphi}_{\varphi(a)}\bar{\varphi}_{\varphi(b)}(\bar{a}_0)=\varphi(a)\circ_2\varphi(b)$, since $\bar{\varphi}_{\varphi(a)}=\varphi_{\varphi(a)}\varphi^{-1}$. Hence, $\varphi(a\circ_1 b)=\varphi(a)\circ_2\varphi(b)$ and so φ is an isomorphism.

Remark 2.8. In this note we have considered communication channels with more than two participants and we have demonstrated that the functioning of such a channel is governed by an abelian group. It has been shown in [2] that for a bidirectional channel consisting of two participants the situation is rather arbitrary. The dynamic labeled 2-structure from [2] consists of a number of nodes (processors) and bidirectional communication channels between *every* two processors. There are a number of ways to generalize such a dynamic labeled 2-structure so that processors would be connected by hyperedges rather than edges. In any case one would expect the resulting hypergraph to be connected. Then it easily follows from the considerations of this note that (if the hyperedges consist of more than two nodes) the work of such a structure is governed by an abelian group.

Acknowledgment

The research presented here has been carried out within the ESPRIT Basic Research Working Group COMPUGRAPH.

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